The Riemann-Hilbert method: from Toeplitz operators to black holes

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Encontro de Ciência 2017

Lisboa, 3-5 Julho

What is a Riemann-Hilbert problem?

Problem: To determine φ such that

$$\Delta\varphi = 0 \quad \text{in} \quad \mathbb{D}\,, \quad \varphi \ \text{continuous} \quad \text{on } \mathbb{T}$$
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$$\begin{gathered}\phi_+(t) + \phi_-(t) = f(t)\,, \;\; t \in \mathbb{T} \\ \Leftrightarrow \; -\phi_+ = \phi_- - f\,, \text{ on } \mathbb{T} \\ g\,\phi_+ = \phi_- - f\,, \text{ on } \mathbb{T}\end{gathered}$$

$$\underbrace{g}_{n\times n}\ \underbrace{\phi_+}_{n\times 1} = \underbrace{\phi_-}_{n\times 1} + \underbrace{f}_{n\times 1} \qquad \text{vectorial RHP}$$

$$\underbrace{g}_{n\times n}\ \underbrace{M_+}_{n\times n} = \underbrace{M_-}_{n\times n} \qquad \text{matrix (factorization) RHP}$$

Bounded Wiener-Hopf (or Birkhoff) factorization:

$g = M_- M_+^{-1}$ $\qquad$ $M_+^{\pm 1}$ analytic and bounded in $\mathbb{D}$

$M_-^{\pm 1}$ analytic and bounded in $\mathbb{C} \setminus \mathit{clos}\mathbb{D}$

RH approach: to reduce a problem to the reconstruction of a function analytic in $\mathbb{C} \setminus \Gamma$ from jump conditions across Γ.

Applications in

- Diffraction problems
- Elastodynamics
- Singular integral equations
- Combinatorial probability
- Random matrices
- Orthogonal polynomials
- Integrable systems

Deift, Its, Kapaev, Novokshenov, Fokas, Ablowitz, Bleher, Östensson

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The Riemann-Hilbert method: a Swiss Army knife
Marko Bertola (Concordia University), 2012

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Considerable progress has been made in **explicit factorisation methods**.

M. C. Câmara, A. F. dos Santos and P. F. dos Santos: *Matrix Riemann-Hilbert problems and factorization on Riemann surfaces*, J. Funct. Anal. (2008).

M. C. Câmara, C. Diogo and L. Rodman: *Fredholmness of Toeplitz operators and corona Problems*, J. Funct. Anal. (2010).

M. C. Câmara, C. Diogo, Yu. Karlovich and I.M. Spitkovsky: *Factorizations, Riemann-Hilbert problems and the corona theorem*, J. London Math. Soc. 86 (2012).

M. C. Câmara, C. Diogo and I.M. Spitkovsky: *Toeplitz operators of finite interval type and the table method*, J. Math. Anal. Appl. (2014).

Spectral properties and kernels of Toeplitz operators (TO)

In the context of $L^2(\mathbb{T})$ or $L^2(\mathbb{R})$:

$$L^2 = H_+^2 \oplus H_2^- \qquad\qquad H_2^{\pm} = \mathcal{F} L^2(\mathbb{R}^{\pm})$$

$$P^+ : L^2 \longrightarrow H_2^+$$

Toeplitz operator:
$T_g : H_2^+ \longrightarrow H_2^+ \qquad g \in L^\infty$
$T_g\, \varphi_+ = P^+ g\, \varphi_+ \qquad$ g is the **symbol** of the TO (it can be matricial).

In $L^2(\mathbb{R})$, TO are unitarily equivalent, via the Fourier transform, to convolution operators on the half-line $\mathbb{R}^+$.

Toeplitz operators are intimately related to RHP:

1. Fredholmness, invertibility, the dimension of the kernel and the cokernel (and therefore their spectral properties) are determined by a RH factorisation of their symbol.

   In particular: T_g is invertible $\iff g = M_- M_+^{-1}$

   Progress in developing methods to explicitly solve **RH factorisation problems** goes hand in hand with progress in the **spectral theory of Toeplitz operators**

2. Kernels of TO:
   many important spaces of functions, such as **model spaces**, can be described as Toeplitz kernels.
   Toeplitz kernels consist of the solutions to a vectorial RHP

$$g\phi_+ = \phi_-$$

Some recent results taking this RH approach to Toeplitz kernels:

- new (and surprising) properties of all Toeplitz kernels

  M. C. Câmara and J. R. Partington, *Near invariance and kernels of Toeplitz operators*, J. Anal. Math. (2014).

- generalisation of Hitt's and Hayashi's results on nearly invariant subspaces to a Banach space setting

  M.C. Câmara and J.R. Partington, *Finite-dimensional Toeplitz kernels and nearly-invariant subspaces*, J. Operator Theory (2016).

- characterisation of the multipliers between Toeplitz kernels

  M.C. Câmara and J.R. Partington, *Multipliers between Toeplitz kernels* (2017).

Constructing new solutions to Einstein's field equations

- Einstein's field equations are **nonlinear** PDE's.
- They are very difficult to solve in general, so one must concentrate on special classes of solutions which exhibit symmetries.
- Reduced (2 dimensional) field equations:

$$d(\rho * A) = 0.$$

Here $*$ denotes the Hodge dual ($*d\rho = dv$, $*dv = -d\rho$) and A is a matrix one-form $A = M^{-1}dM$ where $M(\rho, v)$ determines the solution to Einstein's equations.

- Following Breitenlohner-Maison's approach (1987) one can construct a **linear** system depending on an additional variable called the **spectral parameter** τ:

$$\tau(dX + AX) = *dX.$$

$$d(\rho * A) = 0 \qquad A = M^{-1} dM \tag{3}$$

$$\tau(dX + AX) = *dX \qquad \textbf{Lax pair} \tag{4}$$

(3) is a compatibility condition for (4) (**integrable system**) provided that τ satisfies a certain differential equation

$$\implies \tau\omega = \tau v + \frac{\rho}{2}(1 - \tau^2) \qquad \textbf{spectral curve}$$

The **RH approach** ([CCMN]): to each solution of (3) we can associate a matrix $\mathcal{M}(\omega)$ such that

$$\mathcal{M}(\omega)\big|_{\omega = v + \frac{\rho}{2}\frac{1-\tau^2}{\tau}} = M_-(\tau) M_+^{-1}(\tau) \qquad |\tau| = 1$$

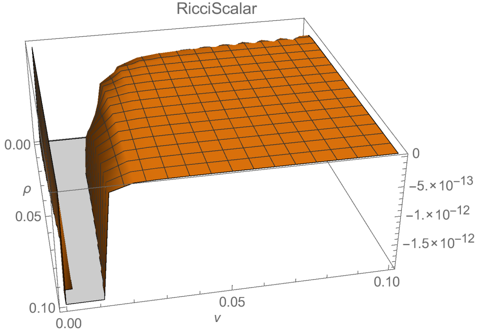
$$M_+^{-1} = X, \qquad \lim_{\tau \to \infty} M_-(\tau) = M(\rho, v)$$

and conversely.

Using explicit factorisation methods:

- we solve the long-standing problem of **constructing extremal black holes** within the RH formulation;
- by acting with elements of the Geroch group on the monodromy matrix of a "seed solution", we obtain **new solutions** to Einstein's equations.

A concrete example: Deforming the monodromy matrix corresponding to the solution that describes the near horizon region of an extremal black hole, we obtain a new solution. It is **explicit, completely regular**, and exhibits **unexpected properties that one would not guess a priori**.



[CCMN] M.C. Câmara, G.L. Cardoso, T. Mohaupt, S. Nampuri: *A Riemann-Hilbert approach to rotating attractors* , JHEP (2017).