Imaginary time in geometry and quantization

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Plan of the talk

- Flows on manifolds
- Imaginary time
- Applications I: Kähler geometry
- Applications II: geometric quantization

Flows on manifolds

Let M be a compact manifold. Recall that the flow of a smooth vector field X on M defines a one-parameter group of diffeomorphisms

$$\varphi_t : M \to M, \quad t \in \mathbb{R},$$

with $\varphi_s \circ \varphi_t = \varphi_{s+t}$. This is a consequence of the Picard-Lindelöf theorem for the existence, uniqueness and smooth dependence on initial conditions for solutions of ODEs.

If you're lost already: think of a surface in $\mathbb{R}^3$ with a smooth distribution of tangential velocities and a fluid moving in time over the surface with that velocity pattern.

Suppose now that M is real analytic (that is, we can cover it by an atlas such that coordinate transformations are power series) and that X is real analytic (that is, it has real analytic components in the above coordinate systems). Then, from the theory of ODEs it follows that the flow φ_t will also be real analytic (that is, its local expressions in the above coordinate systems will be real analytic).

Moreover, φ_t will be real analytic in the variable t. Then, for instance, if $f \in C^\omega(M)$ we will have

$$\varphi_t^* f = f \circ \varphi_t = e^{tX} \cdot f,$$

where in the power series X acts on functions as a first-order differential operator, as usual. Note that, in this real analytic context, this expression is an *actual convergent* power series, not just a convenient "formal" notation.

The *Lie series* e^{tX} acts as an automorphism of the algebra of real analytic functions:

$$e^{tX}(fg) = \left(e^{tX} f\right)\left(e^{tX} g\right).$$

Imaginary time

Suppose now that (M, J) is a complex manifold, that is it can be covered by local holomorphic charts with holomorphic coordinate transformations. Take a local system of holomorphic coordinates $\{z_j\}_{j=1,\ldots,n=\dim_{\mathbb{C}} M}$ around $p \in M$. Since a convergent power series in the real variable t always has a radius of convergence in the complex plane, it follows that there exists $T > 0$ so that we can analytically continue the Lie series in t and define *new* coordinates

$$z_j^\tau = e^{\tau X} z_j, \ \ j = 1, \ldots, n,$$

on a neighboorhood of p, where $\tau \in \mathbb{C}$ and $|\tau| < T$. Note that the radius of convergence depends on p but since the coefficients of the powers series are real analytic in $z, \bar{z}$, the above local lower bound $T > 0$ exists [Grobner 1967].

Since the Lie series acts as an automorphism of the algebra of real analytic functions, it will preserve the holomorphic coordinate transformations on M. Compactness of M will then give:

Theorem: [Mourão-N, 2015] There exists $T > 0$ such that for $\tau \in \mathbb{C}$ with $|\tau| < T$, the above action of the Lie series on coordinates defines a global diffeomorphism of M, φ_τ, and a new complex structure J_τ such that

$$\varphi_\tau : (M, J_\tau) \to (M, J)$$

is a biholomorphism.

We now wish to consider the case when (M, ω) is a symplectic manifold.

For the non-expert: on a symplectic manifold, a smooth function $H \in C^\infty(M)$ defines an Hamiltonian vector field X_H whose flow dynamics generalizes Hamilton's equations from classical mechanics:

$$\begin{cases} \dot{q}_j & = & \frac{\partial H}{\partial p_j} \\ \dot{p}_j & = & -\frac{\partial H}{\partial q_j} \end{cases}$$

in local coordinates (p, q).

In this case, that is for $X = X_H$, the diffeomorphisms $\varphi_t, t \in \mathbb{R}$, will be symplectomorphisms, so that $\varphi_t^* \omega = \omega$.

But, crucially, this will no longer happen in imaginary time,

$$\varphi_\tau^* \omega \neq \omega,$$

for $Im\,(\tau) \neq 0$, in general, when such φ_τ can be defined.

Warning: $\varphi_\tau \circ \varphi_{\tau'} \neq \varphi_{\tau+\tau'}$ in general! By its very definition, φ_τ *depends* on J.

The perfect symbiosis between complex and symplectic structure occurs for a Kähler manifold (M, ω, J, γ), where γ is a Riemannian metric and both ω, γ are appropriately compatible with J. Assume M is compact. For an Hamiltonian function $H \in C^\omega(M)$ and for the corresponding Hamiltonian vector field X_H, there will then exist $T > 0$ and well-defined diffeomorphisms

$$\varphi_\tau : M \to M, \;\; \tau \in \mathbb{C}, \;\; |\tau| < T.$$

It follows from the properties of ω in the Kähler setting (namely that it is a $(1,1)$-form) that the new complex structure J_τ is still compatible with the original ω. So, we get a new Kähler structure $(M, \omega, J_\tau, \gamma_\tau)$.

Theorem: [Mourão-N, 2015] For $\tau \in \mathbb{C}$, $|\tau| < T$, $(M, \omega, J_\tau, \gamma_\tau)$ is a Kähler manifold (with a new Riemannian metric γ_τ). There exists a reasonably explicit formula for the Kähler potential.

Sometimes, these results hold even if M is not compact and for $T = +\infty$.

Applications I: Kähler geometry

The space of Kähler forms in the class $[\omega] \in H^{1,1}(M)$ is

$$\mathcal{H} = \{\phi \in C^\infty(M) : \omega_\phi = \omega + i\partial\bar{\partial}\phi > 0\}.$$

The space of Kähler metrics in the class $[\omega]$ is then given by $\mathcal{H}/\mathbb{R}$.

This space can be equipped with the Donaldson-Mabuchi-Semmes metric where

$$||\delta\phi||_\phi^2 = \int_M (\delta\phi)^2 d\mu_\phi, \;\; d\mu_\phi = \frac{1}{n!}\omega_\phi^n.$$

Theorem: [Mourão-N 2015] The family of Kähler metrics γ_τ is a geodesic family with respect to the Mabuchi metric.

These geodesics play a prominent role in recent work on the relation between algebro-geometric stability properties of M and the existence of Kähler metrics with constant scalar curvature on it. [Chen, Donaldson, Sun, Tian, Rubistein, Zelditch, etc.]

Applications II: geometric quantization

Morally, the process of "quantization" of a symplectic manifold (M, ω) should assign to it a Hilbert space $\mathcal{H}$, such that functions $f \in C^\infty(M)$ are promoted to operators $\widehat{f}$ acting on $\mathcal{H}$ with

$$\widehat{\{f, g\}}_{\text{P.B.}} = \frac{i}{\hbar}[\widehat{f}, \widehat{g}], \;\; f, g \in C^\infty(M),$$

along with a few other natural conditions including the irreducibility of this representation $\mathcal{H}$. It is known that this problem has no solution if one imposes all these requirements.

Geometric quantization is a rich framework where one can study mathematical issues related to the problem of quantization.

The assignment of $\mathcal{H}$ to (M, ω) depends on choices (on the choice of a *polarization*) and the most fundamental problem in geometric quantization is understanding if the Hilbert spaces for different choices are - or not - unitarily equivalent in a natural way.

For a Kähler manifold (M, ω, J, γ), $\mathcal{H}_J$ can be obtained from specific holomorphic data of algebro-geometric flavour.

The diffeomorphisms φ_τ map polarizations to polarizations, since they are generated by Hamiltonian vector fields. It is tremendously interesting to study the quantizations along the geodesic families of Kähler structures $(M, \omega, J_\tau, \gamma_\tau)$. This is very successful in rich families of interesting symplectic manifolds: cotangent bunldes of compact Lie groups, abelian varieties, toric manifolds. For instance [Baier-Florentino-Mourão-N, 2011; Kirwin-Mourão-N, 2013].

In some cases, interesting Gromov-Hausdorff metric collapse and tropical geometry occur as $\tau \to \infty$. [Baier-Florentino-Mourão-N, 2011].

In some cases, singular Hamiltonian functions lead to interesting effects such as metrics with cone-angle singularities [Kirwin-Mourão-N, 2016].

THANK YOU.